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Helicoidal surfaces under the cubic screw motion in Minkowski 3-space

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Abstract

In this paper, we construct helicoidal surfaces under the cubic screw motion with prescribed mean or Gauss curvature in Minkowski 3-space E_1^3 . We also find explicitly the relation between the mean curvature and Gauss curvature of them. Furthermore, we discuss helicoidal surfaces under the cubic screw motion with $H^2 = K$ and prove that these surfaces have equal constant principal curvatures. © 2005 Elsevier Inc. All rights reserved.

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1. Introduction

A helicoidal surface in R^3 is defined as the orbit of a plane curve under a screw motion. As for helicoidal surfaces in R^3 , the cases with prescribed mean or Gauss curvatures have been studied by Christos Baikoussis and Themis Koufogiorgos [1], and the cases $H = \text{const}$ or $K = \text{const}$ have been solved by Do Carmo and Dajczer [4].

In this paper we will consider helicoidal surfaces in *Minkowski 3-space* E_1^3 with the indefinite metric $ds^2 = -dx_1^2 + dx_2^2 + dx_3^2$. Many of the classical results from Euclid-

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can geometry have a Minkowski counterpart, like the existence of Delaunay surface. But because of the difference between their metrics, many essential concepts in Minkowski space such as the vectors, the frames, the motions of particles change a lot. Especially the presence of null vectors causes important differences.

A Lorentzian screw motion was defined to be a Lorentzian rotation around an axis together with a translation in the direction of the axis. Rotational surfaces and helicoidal surfaces under a Lorentzian screw motion with prescribed mean or Gauss curvatures have been studied by Beneki et al. [2,3] and the authors [6].

Dillen and Kühnel [5] pointed out that a Lorentzian rotation around a null axis, together with a translation in the direction of the axis, is again a Lorentzian rotation around a null axis (see Remark 2.1 below), and there exist other non-trivial 1-parameter families of translations that, together with a Lorentzian rotation around a null axis, constitute a 1-parameter group of Lorentzian motions, the so-called *cubic screw motion* [5], which is expressed as:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1 + \frac{s^2}{2} & -\frac{s^2}{2} & s \\ \frac{s^2}{2} & 1 - \frac{s^2}{2} & s \\ s & -s & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + h \begin{pmatrix} \frac{s^3}{3} + s \\ \frac{s^3}{3} - s \\ s^2 \end{pmatrix}.$$

Obviously, a cubic screw motion is different from a non-cubic case. A non-cubic screw motion has the property that, if we take a point of the axis, then the orbit of that point is simply the axis (or the point itself if the screw motion is a rotation). A cubic screw motion does not have that property. In fact, the orbit of the origin under a cubic screw motion is a cubic null curve, just given by the translational part of the cubic screw motion.

In this paper, we study two kinds of helicoidal surfaces under the cubic screw motion. The main technique we apply is the pseudo-orthonormal frame method. By several special changes of variables, we solve a series of ODEs and construct helicoidal surfaces under the cubic screw motion with prescribed mean or Gauss curvatures (see Theorems 3.1, 3.2, 4.1, 4.2). Then we study the geometrical properties of them and find explicitly the relation between the mean curvature and Gauss curvature (see Propositions 3.1, 4.1). Furthermore, we discuss the helicoidal surfaces under the cubic screw motion with $H^2 = K$ and prove that they are surfaces with equal constant principal curvatures (see Theorems 3.3, 4.3).

2. Helicoidal surfaces under a cubic screw motion in E_1^3

We denote by E_1^3 the 3-dimensional Minkowski space with the Lorentz metric

$$g(x, y) = -x_1 y_1 + x_2 y_2 + x_3 y_3,$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$. An Lorentzian transformation of E_1^3 is a linear map that preserves the bilinear form g .

Now, we consider a pseudo-orthonormal basis of E_1^3 , i.e., a basis $\{e_1, e_2, e_3\}$ such that

$$g(e_1, e_1) = g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_3) = 0, \quad g(e_1, e_3) = g(e_2, e_2) = 1.$$

Then the Lorentz metric can be expressed as

$$g(x, x) = 2x_1 x_3 + x_2^2, \quad x = \sum x_k e_k.$$

Let

$$(e_1, e_2, e_3) = (\eta_1, \eta_2, \eta_3)X,$$

where $\{\eta_1, \eta_2, \eta_3\}$ is an orthonormal basis such that

$$g(\eta_i, \eta_j) = \varepsilon \delta_{ij}, \quad \varepsilon = \begin{cases} -1, & \text{if } i = 1, \\ 1, & \text{if } i = 2, 3, \end{cases}$$

and

$$X = \begin{pmatrix} -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \end{pmatrix}.$$

Then the cubic screw motion around the axis e_3 can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto X^{-1} \begin{pmatrix} 1 + \frac{s^2}{2} & -\frac{s^2}{2} & s \\ \frac{s^2}{2} & 1 - \frac{s^2}{2} & s \\ s & -s & 1 \end{pmatrix} X \begin{pmatrix} x \\ y \\ z \end{pmatrix} + hX^{-1} \begin{pmatrix} \frac{s^3}{3} + s \\ \frac{s^3}{3} - s \\ s^2 \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto A(v) \begin{pmatrix} x \\ y \\ z \end{pmatrix} + h\beta(v), \quad (2.1)$$

where

$$A(v) = \begin{pmatrix} 1 & 0 & 0 \\ v & 1 & 0 \\ -\frac{v^2}{2} & -v & 1 \end{pmatrix}, \quad \beta(v) = \begin{pmatrix} v \\ \frac{v^2}{2} \\ -\frac{v^3}{6} \end{pmatrix}, \quad v = -\sqrt{2}s.$$

Remark 2.1. Incidentally, it is easy to see that

$$A(v) \begin{pmatrix} x \\ y \\ z \end{pmatrix} + h \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} = A(v) \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix} \right) + \begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix}.$$

This indicates that a rotation around the null coordinate axis Oe_3 , together with a translation in the direction of Oe_3 , is again a rotation around the null line l :

$$\frac{x}{0} = \frac{y-h}{0} = \frac{z}{1}.$$

Hence we have confirmed the statement mentioned in the introduction.

Definition 2.1. Let $\gamma : I = (a, b) \subset \mathbb{R} \rightarrow P$ be a curve in a plane P in E_1^3 and denote by L a straight line that does not intersect the curve γ . A helicoidal surface under a cubic screw motion in E_1^3 is defined as a non-degenerate surface that is generated by a cubic screw motion around L .

We distinguish the following two cases:

Case 1. Let $\gamma_1(u) = (u, 0, f(u))$, $u > 0$ be a curve in the Oe_1e_3 plane. Suppose that S_1 be the helicoidal surface generated by $\gamma_1(u)$ under a cubic screw motion with pitch h , the position vector r of which has the form

$$r(u, v) = \left(u + hv, uv + \frac{hv^2}{2}, f(u) - \frac{uv^2}{2} - \frac{hv^3}{6} \right). \quad (2.2)$$

We call S_1 a helicoidal surface of type I.

Case 2. Let $\gamma_2(u) = (0, u, f(u))$, $u > 0$ be a curve in the Oe_2e_3 plane. Suppose that S_2 be the helicoidal surface generated by $\gamma_2(u)$ under a cubic screw motion with pitch h , the position vector r of which has the form

$$r(u, v) = \left(hv, u + \frac{hv^2}{2}, f(u) - uv - \frac{hv^3}{6} \right). \quad (2.3)$$

We call S_2 a helicoidal surface of type II.

We say that a surface in E_1^3 is of type I^+ or type I^- (respectively type II^+ or type II^-) if the discriminant $D = EG - F^2$ is positive or negative, where E, F, G are the coefficients of its first fundamental form.

3. Helicoidal surface of type I

Let S_1 be the helicoidal surface given by (2.2). The first fundamental form I , the second fundamental form II of S_1 are

$$I = 2f' du^2 + 2hf' du dv + u^2 dv^2$$

and

$$II = |D|^{-1/2} [uf'' du^2 + 2hf' du dv + (h^2 f' - u^2) dv^2],$$

where

$$D = EG - F^2 = 2u^2 f' - h^2 f'^2$$

and the *prime* denotes derivative with respect to u .

By a direct computation, we can see that the mean curvature H and the Gauss curvature K of S_1 are given by

$$H = \frac{u^3 f'' - 2u^2 f'}{2D|D|^{1/2}} \quad (3.1)$$

and

$$K = \frac{-u^3 f'' + h^2 u f' f'' - h^2 f'^2}{D|D|}. \quad (3.2)$$

Remark 3.1. From (3.1) and (3.2) we can see that H and K are independent of the parameter v .

3.1. The solution of Eq. (3.1)

Let

$$A = (u^2 - h^2 f')|D|^{-1/2}. \quad (3.3)$$

For $u \in (0, +\infty)$, from (3.1), we have

$$H = -\frac{1}{2u}(A)'. \quad (3.4)$$

Combining (3.3) and (3.4), we get the general solution of (3.1),

$$f = \int \frac{u^2}{h^2} [1 \pm |A|(A^2 + \varepsilon h^2)^{-1/2}] du + c_2, \quad (3.5)$$

where $A = -(\int H du^2 + c_1)$ and

$$\varepsilon = \begin{cases} 1, & \text{if } S_1 \text{ is space-like,} \\ -1, & \text{if } S_1 \text{ is time-like,} \end{cases}$$

and c_1, c_2 are constants of integration.

Now, suppose that h is a given non-zero real constant and $H(u)$ is a real-valued smooth function defined on an open interval $I \subset (0, +\infty)$. Let

$$F(u, c_1) = \left(\int H du^2 + c_1 \right)^2 + \varepsilon h^2$$

be a function defined on $I \times R \subset R^2$. For any $u_0 \in I$, denote

$$c'_1 = 1 + |h| - \left(\int H du^2 \right)(u_0).$$

Consequently, we can find an open subinterval I' of I containing u_0 and an open interval B of R containing c'_1 such that the function $F(u, c_1)$ is positive for any $(u, c_1) \in I' \times B$. In fact, we have $F(u_0, c'_1) > 1 > 0$. Then by the continuity of F , it is positive in a subset of $I' \times B \subset R^2$ of the form.

Given a non-zero constant h and a real-valued function $H(u)$. For any $(u, c_1) \in I' \times B$ and $c_2 \in R$, we define a two-parametric family of curves

$$\gamma(u, H(u), h, c_1, c_2) = \left(u, 0, \int \frac{u^2}{h^2} [1 \pm |A|(A^2 + \varepsilon h^2)^{-1/2}] du + c_2 \right),$$

where $A = -(\int H du^2 + c_1)$.

Applying a cubic screw motion of pitch h on these curves, we get a two-parametric family of helicoidal surfaces of type I^+ (respectively I^-) with mean curvature $H(u)$, $u \in I'$ and pitch h . So we have proved the following:

Theorem 3.1. *Let $[O; e_1, e_2, e_3]$ be the pseudo-orthogonal frame with e_3 as a null vector. Suppose that h is a given non-zero real constant and $H(u)$ is a real-valued smooth*

function defined on an open interval I . Then, for any $u_0 \in I$ there exist an open subinterval I' of I containing u_0 and a two-parametric family of helicoidal surfaces of type I^+ (respectively I^-) generated by the two-parametric family of plane curves

$$\gamma(u, H(u), h, c_1, c_2) = \left(u, 0, \int \frac{u^2}{h^2} [1 \pm |A|(A^2 + \varepsilon h^2)^{-1/2}] du + c_2 \right),$$

where $A = -(\int H u du^2 + c_1)$, under a cubic screw motion with mean curvature $H(u)$ and pitch h .

Remark 3.2. From (3.1) we can easily see that a helicoidal surface under a cubic screw motion of type I^+ (respectively I^-) with zero mean curvature is independent of its pitch h .

3.2. The solution of Eq. (3.2)

Let

$$B = u^2 |D|^{-1}. \quad (3.6)$$

For $u \in (0, +\infty)$, from (3.2), we have

$$K = \frac{1}{2u} B'. \quad (3.7)$$

Combining (3.6), (3.7), we get the general solution of (3.2),

$$f = \int \frac{u^2}{h^2} [1 \pm (1 - \varepsilon h^2 u^{-2} B^{-1})^{1/2}] du + c_2, \quad (3.8)$$

where $B = (\int K du^2 + c_1)$ and

$$\varepsilon = \begin{cases} 1, & \text{if } S_1 \text{ is space-like,} \\ -1, & \text{if } S_1 \text{ is time-like,} \end{cases}$$

and c_1, c_2 are constants of integration.

Suppose that h is a given non-zero real constant and $K(u)$ is a smooth real-valued smooth function defined on an open interval $I \subset (0, +\infty)$. Let

$$F_1(u, c_1) = \int K du^2 + c_1, \quad F_2(u, c_1) = 1 - \varepsilon h^2 u^{-2} \left(\int K du^2 + c_1 \right)^{-1}$$

be two functions defined on $I \times \mathbb{R} \subset \mathbb{R}^2$.

For any $u_0 \in I$, denote

$$c'_1 = 2h^2/u_0^2 - \left(\int K du^2 \right)(u_0).$$

Consequently, we can find an open subinterval I' of I containing u_0 and an open interval U of \mathbb{R} containing c'_1 such that F_1 and F_2 are positive for any $(u, c_1) \in I' \times U$. In fact, we have $F_1(u_0, c'_1) = 2h^2/u_0^2 > 0$ and $F_2(u_0, c'_1) = 1/2 > 0$. By the continuity of F_1, F_2 , they are positive in $I' \times U \subset \mathbb{R}^2$.

For any $(u, c_1) \in I' \times U$, $c_2 \in \mathbb{R}$, a given non-zero constant h and a given real-valued function $H(u)$, we can define a two-parametric family of curves

$$\gamma(u, K(u), h, c_1, c_2) = \left(u, 0, \int \frac{u^2}{h^2} [1 \pm (1 - \varepsilon h^2 u^{-2} B^{-1})^{1/2}] du + c_2 \right),$$

where $B = (\int K du^2 + c_1)$.

Applying a cubic screw motion of pitch h on these curves, we get a two-parametric family helicoidal surfaces of type I^+ (respectively I^-) around e_3 with Gauss curvature $K(u)$ ($u \in I'$) and pitch h . So we have proved the following:

Theorem 3.2. *Let $[O; e_1, e_2, e_3]$ be the pseudo-orthogonal frame with e_3 as a null vector. Suppose that h is a given non-zero real constant and $K(u)$ is a real-valued smooth function defined on an open interval I . Then, for any $u_0 \in I$ there exist an open subinterval I' of I containing u_0 and a two-parametric family of helicoidal surfaces of type I^+ (respectively I^-) generated by the two-parametric family of plane curves*

$$\gamma(u, K(u), h, c_1, c_2) = \left(u, 0, \int \frac{u^2}{h^2} [1 \pm (1 - \varepsilon h^2 u^{-2} B^{-1})^{1/2}] du + c_2 \right),$$

where $B = (\int K du^2 + c_1)$, under a cubic screw motion with Gauss curvature $K(u)$ and pitch h .

Obviously, any helicoidal surface is a Weingarten surface because its mean curvature H and Gauss curvature K depend on only on one parameter. For a helicoidal surface of type I , we obtain an equation indicating explicitly the relation between its mean curvature and Gauss curvature.

For any $u_0 \in (0, +\infty)$, we find two constants l_1, l_2 and an open interval $I \subset (0, +\infty)$ containing u_0 such that

$$\left(\int_{u_0}^u H du^2 + l_1 \right)^2 + \varepsilon h^2, \quad \int_{u_0}^u K du^2 + l_2 \quad \text{and} \\ 1 - \varepsilon h^2 u^{-2} \left(\int_{u_0}^u K du^2 + l_2 \right)^{-1}$$

are all positive in I . Denote

$$\mathcal{H}(u) = \int_{u_0}^u H du^2 + l_1 \quad \text{and} \quad \mathcal{K}(u) = \int_{u_0}^u K du^2 + l_2. \quad (3.9)$$

Then from (3.5) and (3.8), we get the following proposition.

Proposition 3.1. *For any $u \in I$, we have*

$$u^2 \mathcal{K} - \mathcal{H}^2 = \varepsilon h^2, \quad (3.10)$$

where \mathcal{H} and \mathcal{K} are given by (3.9).

Denote $w = u^2$. Substituting $H^2 = K$ into (3.10) and taking the derivative with respect to w on both sides twice, we have

$$H' = 0 \quad \text{or} \quad wH - \mathcal{H} = 0.$$

Taking the derivative with respect to w on both sides of the second equation with respect to w , we get $H' = 0$ again. This indicates that $H = \text{const}$ and $K = \text{const}$. So we get

Theorem 3.3. *Helicoidal surfaces of type I with $H^2 = K$ have equal constant principal curvatures.*

Remark 3.3. There exist helicoidal surfaces with equal constant principal curvatures (see Example 4.1, 4.2). On the other hand, Dillen and Kühnel [5] proved that any ruled surface $r(s) + tX(s)$ with a null ruling X in E_1^3 satisfied $H^2 = K = 1/F^2$. Here $F = \langle r', X \rangle$ and it is an function with respect to s in general. So there also exist surfaces satisfying $H^2 = K \neq \text{const}$ in E_1^3 .

4. Helicoidal surface of type II

Let S_2 be the helicoidal surface given by (2.3). The first fundamental form I , the second fundamental form II of S_2 are

$$I = du^2 + 2hf' du dv - 2hu dv^2$$

and

$$II = |D|^{-1/2} (-hf'' du^2 + 2h du dv + h^2 f' dv^2),$$

where

$$D = EG - F^2 = -2hu - h^2 f'^2$$

and the prime denotes derivative with respect to u .

By a direct computation, we can see that the mean curvature H and the Gauss curvature K of S_2 are given by

$$H = \frac{h^2(2uf'' - f')}{2D|D|^{1/2}} \quad (4.1)$$

and

$$K = \frac{-h^2(hf'f'' + 1)}{D|D|}. \quad (4.2)$$

Remark 4.1. From (4.1) and (4.2) we can see that H and K are independent of the parameter v .

4.1. The solution of Eq. (4.1)

Let

$$A = f'|D|^{-1/2}. \quad (4.3)$$

For $u \in (0, +\infty)$, from (4.1), we have

$$H = -\frac{h}{2}A'. \quad (4.4)$$

Combining (4.3) and (4.4), we get the general solution of (4.1)

$$f = \pm \int \left[\frac{-8u(\int H du + c_1)^2}{4h(\int H du + c_1)^2 + \varepsilon h} \right]^{1/2} du + c_2, \quad (4.5)$$

where

$$\varepsilon = \begin{cases} 1, & \text{if } S_1 \text{ is space-like,} \\ -1, & \text{if } S_1 \text{ is time-like,} \end{cases}$$

and c_1, c_2 are constants of integration.

Suppose that h is a given non-zero constant and $H(u)$ is a real-valued smooth function defined on an open interval $I \subset (0, +\infty)$. Let

$$F(u, c_1) = -4h \left(\int H du + c_1 \right)^2 - \varepsilon h$$

be a function defined on $I \times R \subset R^2$.

For any $u_0 \in I$, denote

$$c'_1 = \begin{cases} -(\int H du)(u_0), & \text{if } h > 0, \\ 1 - (\int H du)(u_0), & \text{if } h < 0. \end{cases}$$

Consequently, for any $u_0 \in I$ we can find an open subinterval I' of I containing u_0 and an open interval B of R containing c'_1 such that the function $F(u, c_1)$ is positive for any $(u, c_1) \in I' \times B$. In fact, if $h > 0$, we have $\varepsilon = -1$. Therefore,

$$F(u_0, c'_1) = \begin{cases} h > 0, & \text{if } h > 0, \\ -(4 + \varepsilon)h > 0, & \text{if } h < 0. \end{cases}$$

Then by the continuity of F , it is positive in $I' \times B \subset R^2$.

For any $(u, c_1) \in I' \times B$, $c_2 \in R$, a given non-zero constant h and a given real-valued function $H(u)$ we can define a two-parametric family of curves

$$\gamma(u, H(u), h, c_1, c_2) = \left(0, u, \pm \int \left[\frac{-8u(\int H du + c_1)^2}{4h(\int H du + c_1)^2 + \varepsilon h} \right]^{1/2} du + c_2 \right).$$

Applying a cubic screw motion of pitch h on these curves, we get a two-parametric family of helicoidal surfaces of type II^+ (respectively II^-) with mean curvature $H(u)$, $u \in I'$, and pitch h . So we have proved the following:

Theorem 4.1. Let $[O; e_1, e_2, e_3]$ be the pseudo-orthogonal frame with e_3 as a null vector. Suppose that h is a given non-zero real constant and $H(u)$ is a real-valued smooth function defined on an open interval I . Then, for any $u_0 \in I$ there exist an open subinterval I' of I containing u_0 and a two-parametric family of helicoidal surfaces of type II^+ (respectively II^-) generated by the two-parametric family of plane curves

$$\gamma(u, H(u), h, c_1, c_2) = \left(0, u, \pm \int \left[\frac{-8u(\int H du + c_1)^2}{4h(\int H du + c_1)^2 + \varepsilon h} \right]^{1/2} du + c_2 \right)$$

under a cubic screw motion with mean curvature $H(u)$ and pitch h .

Remark 4.2. From (4.1) we can easily see that a helicoidal surface under a cubic screw motion of type II^+ (respectively II^-) with zero mean curvature is independent of its pitch h .

4.2. The solution of Eq. (4.2)

Let

$$B = |D|^{-1}. \quad (4.6)$$

For $u \in (0, +\infty)$, from (4.2), we have

$$K = -\frac{h}{2}B'$$

or

$$B = -\frac{2}{h} \left(\int k du + c_1 \right). \quad (4.7)$$

Combining (4.6), (4.7), we get the general solution of (4.2),

$$f = \pm \int \left[\frac{4u(\int K du + c_1) - \varepsilon}{-2h(\int K du + c_1)} \right]^{1/2} du + c_2, \quad (4.8)$$

where

$$\varepsilon = \begin{cases} 1, & \text{if } S_1 \text{ is space-like,} \\ -1, & \text{if } S_1 \text{ is time-like,} \end{cases}$$

and c_1, c_2 are constants of integration.

Suppose that h is a given non-zero real constant and $K(u)$ is a smooth real-valued smooth function defined on an open interval $I \subset (0, +\infty)$. Let

$$F_1(u, c_1) = -h \left(\int K du^2 + c_1 \right), \quad F_2(u, c_1) = 4u \left(\int K du^2 + c_1 \right) - \varepsilon$$

be two functions defined on $I \times \mathbb{R} \subset \mathbb{R}^2$. For any $u_0 \in I$, denote

$$c'_1 = \begin{cases} -(\int K du + 1/8u)(u_0), & \text{if } h > 0, \\ -(\int K du - 1/u)(u_0), & \text{if } h < 0. \end{cases}$$

Consequently, we can find an open subinterval I' of I containing u_0 and an open interval U of R containing c'_1 such that the functions F_1 and F_2 are positive for any $(u, c_1) \in I' \times U$. In fact, we have

$$F_1(u_0, c'_1) = \begin{cases} 1/(8hu_0) > 0, & \text{if } h > 0, \\ -1/(hu_0) > 0, & \text{if } h < 0, \end{cases}$$

and

$$F_2(u_0, c'_1) = \begin{cases} 1/2 > 0, & \text{if } h > 0, \\ 4 - \varepsilon > 0, & \text{if } h < 0. \end{cases}$$

Then by the continuity of F_1, F_2 , they are positive in $I' \times U \subset R^2$.

For any $(u, c_1) \in I' \times U$, $c_2 \in R$, a given non-zero constant h and a given real-valued function $H(u)$, we can define a two-parametric family of curves

$$\gamma(u, K(u), h, c_1, c_2) = \left(0, u, \pm \int \left[\frac{4u(\int K du + c_1) - \varepsilon}{-2h(\int K du + c_1)} \right]^{1/2} du + c_2 \right).$$

Applying a cubic screw motion of pitch h on these curves, we get a two-parametric family helicoidal surfaces of type II^+ (respectively II^-) with null axis e_3 with Gauss curvature $K(u)$, $u \in I'$, and pitch h . So we have proved the following:

Theorem 4.2. *Let $[O; e_1, e_2, e_3]$ be the pseudo-orthogonal frame with e_3 as a null vector. Suppose that h is a given non-zero real constant and $K(u)$ is a real-valued smooth function defined on an open interval I . Then, for any $u_0 \in I$ there exist an open subinterval I' of I containing u_0 and a two-parametric family of helicoidal surfaces of type II^+ (respectively II^-) generated by the two-parametric family of plane curves*

$$\gamma(u, K(u), h, c_1, c_2) = \left(0, u, \pm \int \left[\frac{4u(\int K du + c_1) - \varepsilon}{-2h(\int K du + c_1)} \right]^{1/2} du + c_2 \right),$$

under a cubic screw motion with Gauss curvature $K(u)$ and pitch h .

Incidentally, for a helicoidal surface of type II , we obtain an equation indicating explicitly the relation between its mean curvature and Gauss curvature.

For any $u_0 \in (0, +\infty)$, we find two constants l_1, l_2 and an open interval $I \subset (0, +\infty)$ containing u_0 such that

$$\begin{aligned} & -4h \left(\int_{u_0}^u H du + l_1 \right)^2 - \varepsilon h, & -h \left(\int_{u_0}^u K du + l_2 \right), \\ & 4u \left(\int_{u_0}^u K du + l_2 \right) - \varepsilon \end{aligned}$$

are all positive in I . Denote

$$\mathcal{H}(u) = \int_{u_0}^u H du + l_1 \quad \text{and} \quad \mathcal{K}(u) = \int_{u_0}^u K du + l_2. \quad (4.9)$$

Then from (4.5) and (4.8), we get the following proposition.

Proposition 4.1. For any $u \in I$, we have

$$u\mathcal{K} - \mathcal{H}^2 = \frac{\varepsilon}{4}, \quad (4.10)$$

where \mathcal{H} and \mathcal{K} are given by (4.9).

Remark 4.3. Obviously, (4.10) is independent of the pitch h . This is an interesting feature.

As for helicoidal surfaces of type I , we have

Theorem 4.3. Helicoidal surfaces of type II with $H^2 = K$ have equal constant principal curvatures.

The proof of Theorem 4.3 is similar to Theorem 3.3 and we omit it here.

Example 4.1. A helicoidal surface of type I^- , $H = 1$, $K = 1$,

$$r(u, v) = \left(u + v, uv + \frac{v^2}{2}, \frac{u^3}{3} + \frac{u^2 - 1}{3} \sqrt{u^2 + 2} - \frac{uv^2}{2} - \frac{v^3}{6} \right), \quad u > 0, v \in \mathbb{R}$$

(see Fig. 1).

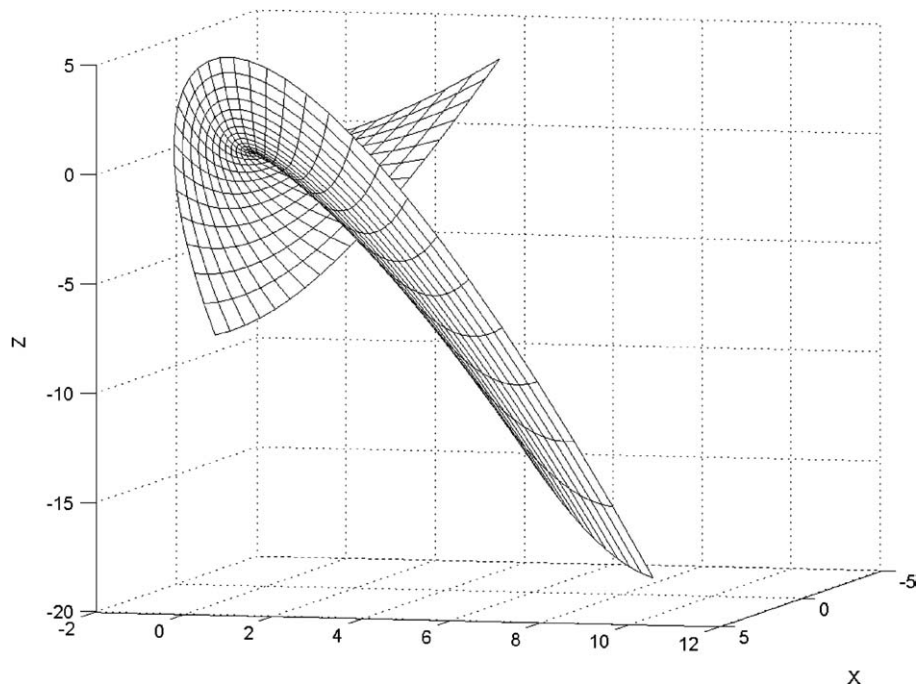


Fig. 1.

Example 4.2. A helicoidal surface of type II^- , $H = -1$, $K = 1$,

$$r(u, v) = \left(v, u + \frac{v^2}{2}, \frac{2u+1}{3}\sqrt{2(1-u)} - uv - \frac{v^3}{6} \right), \quad 0 < u < \frac{1}{2}, v \in \mathbb{R}$$

(see Fig. 2).

Remark 4.4. A surface satisfying $H^2 = K$ in E^3 is an umbilical surface, i.e., a surface whose shape operator has the minimal polynomial $(x - \lambda)$, where λ is its constant principal curvature. But this statement is not true for a surface in E_1^3 . The shape operator of Example 4.2 is written in the basis $\{r_u, r_v\}$ as

$$\begin{pmatrix} -2 & w \\ w^{-1} & 0 \end{pmatrix},$$

where $w = (2 - 2u)^{1/2}$. Obviously, the minimal polynomial of it is $(x + 1)^2$. This indicates that even a surface with equal constant principal curvatures in E_1^3 is not really an umbilical surface.

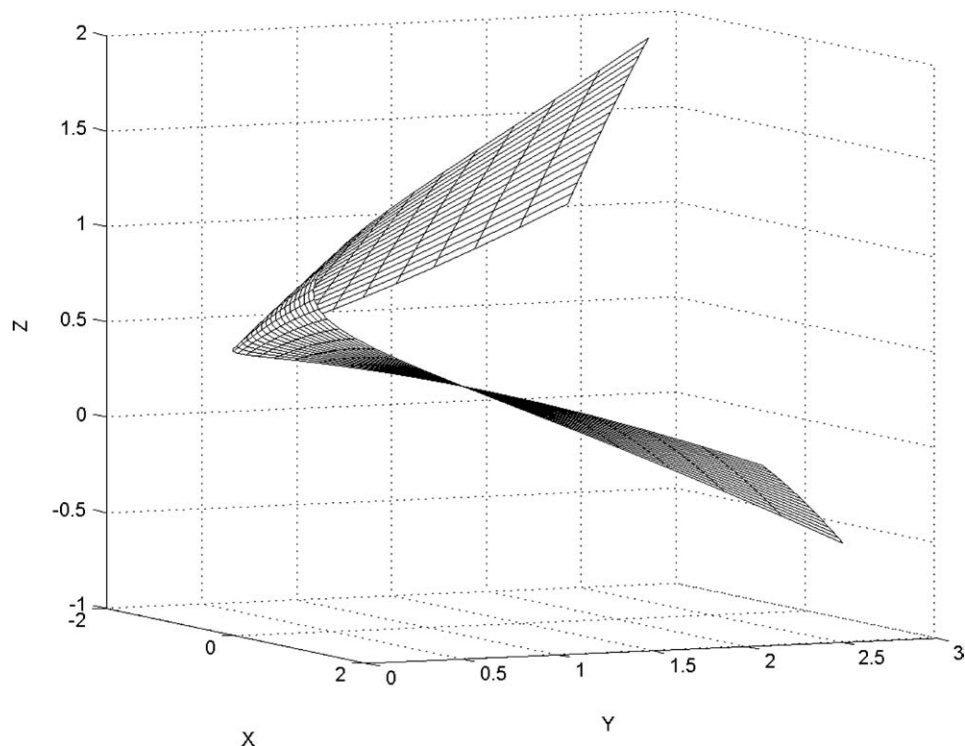


Fig. 2.

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